MATH4341 Topology – Notes #3

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5 General Topological Sets

Definition 5.1. Let X be any set. A **topology** τ on X is a family of subsets of X with the following properties: i) $X \in \tau$; ii) $\phi \in \tau$; iii) if $\{U_\alpha\}_{\alpha \in I}$ is in τ, then so is $\bigcup_{\alpha \in I} U_\alpha$; iv) If $U_1, U_2, \ldots, U_n$ is a finite collection of members of τ, then $\bigcap_{i=1}^{n} U_i$ is also in τ.

We may occasionally write (X, τ) for a topological space.

Example 5.2. i) Any metric space with τ as the topology generated by its metric. If (X, τ), for a topological space, is such that τ is the topology generated by some metric d on X, then (X, τ) is said to be **metrizable**.

ii) Let X be any set and $\tau = \mathcal{P}(X) = \{\text{all subsets of } X\} =$ the powerset of X. Then (X, τ) is a topological space, and τ is said to be the **discrete topology** on X. Evidently, (X, τ) is metrizable.

iii) Let X be any set. The **indiscrete topology** on X is given by $\tau = \{\phi, X\}$.

iv) Let $X = \{a, b\}$. Take $\tau = \{\phi, X, \{a\}\}$. Verify (X, τ) is a topological space. This is the **Sierpinski** or **two-point connected set**.

v) Let X be any set, and let $\tau = \{A \subseteq X : A = \phi \text{ or } X \backslash A \text{ is finite}\}$. Show that τ is indeed a topology (recall ϕ is, by convention, finite). This is the **co-finite topology** on X.

vi) Let X be a non-empty set. Let $p \in X$ be a priori specified point. Then $\tau = \{\phi\} \cup \{A \subseteq X : p \in A\}$ is called a **particular point topology**.

vii) Let X and p be as above. Let $\tau = \{X\} \cup \{A \subseteq X : p \notin A\}$. This is called an **extended point topology**.

Definition 5.3. [1] Let (X, τ) be a topological space. Any member of τ is said to be an **open subset** of X.

Definition 5.4 (Interior of a Set)**.** Le (X, τ) be a topological space, and let $A \subseteq X$. The **interior** of A is by definition $\bigcup U$ over all subsets U of X satisfying i) U is open, and ii) $U \subseteq A$.

Obviously, $\operatorname{int} A$ is the *largest* open set contained in A. Clearly, since arbitrary unions of open sets are open, $\operatorname{int} A$ is open for all subsets $A \subseteq X$. Further A is open if and only if $\operatorname{int} A = A$.

Definition 5.5 (Neighbourhood of a point x)**.** Let (X, τ) be a topological space, and let $x \in X$. Any subset $A \subseteq X$ with $x \in \operatorname{int} A$ is called a **neighborhood of** x ("nbhd").

Definition 5.6 (Closed sets)**.** Let (X, τ) be a topological space. A subset $K \subseteq X$ is closed if $X \backslash K$ is open, i.e., if $X \backslash K \in \tau$.

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[*] Scribed by Matthew R. Dempsky

[1] Reader be warned, herein I diverge from the numbering in Ramakrishna's handouts.

Exercise 5.7. i) ϕ is closed; ii) X is closed; iii) If $K_1, K_2, \ldots, K_e$ are a finite number of closed sets, then $K_1 \cup K_2 \cup \cdots \cup K_e$ is also closed; iv) If $\{K_i\}_{i\in I}$ is a family of closed sets, then so is $\bigcap_{i\in I} K_i$.

Definition 5.8. Let X be a topological space, and let $A \subseteq X$. A point $x \in X$ is a **cluster point of** A if every neighborhood U of x is such that $(U \cap A)\backslash\{x\} \neq \phi$. The **derived set of** A, denoted A', is the set of all cluster points of X.

Remark 5.9. Since $\operatorname{int} B$ is open for all $B \subseteq X$, we see that $x \in X$ is a cluster point of A if and only if every open set C containing x satisfies $C \cap A\backslash\{x\} \neq \phi$. Indeed, if x is a cluster point per Definition 5.8, then since the open set C is a neighborhood of x, we must have $C \cap A\backslash\{x\} \neq \phi$. Conversely, let $x \in X$ be such that for every open set C containing x we have $C \cap A\backslash\{x\} \neq \phi$. Let U be a neighborhood of x. Thus $x \in \operatorname{int} U$. But $\operatorname{int} U$ is open. So we must have $(\operatorname{int} U) \cap A\backslash\{x\} \neq \phi$. Since $\operatorname{int} U \subseteq U$, this implies $U \bigcap A\backslash\{x\} \neq \phi$, i.e., that x is a cluster point per Definition 5.8.

Definition 5.10. The **closure** of a set A is given by $\bigcap F$ where F is closed and $F \supseteq A$. The closure of A is denoted $\overline{A}$.

Remark 5.11. Since arbitrary intersection of closed sets are closed, it is evident that $\overline{A}$ is closed for any $A \subseteq X$. Further, $\overline{A}$ is the smallest closed set containing A.

Theorem 5.12 (Alternative characterizations of closure)**.** *i) $x \in \overline{A}$ if and only if every neighborhood U of x intersects A in a non-empty fashion; ii) $\overline{A} = A \cup A'$.*

Proof. i) Let $x \in \overline{A}$, per Definition 5.10. Let U be a neighborhood of x. If $U \cap A = \phi$, then $X\backslash \operatorname{int} U$ is a closed set, and $A \subseteq X\backslash \operatorname{int} U$. (Indeed, $U \cap A = \phi$ implies $A \subseteq X\backslash U$. Since $\operatorname{int} U \subseteq U$, we must have $A \subseteq X\backslash \operatorname{int} U$.) But then $\overline{A} \subseteq X\backslash \operatorname{int} U$ since $\overline{A}$ is the intersection of all closed sets containing A. But then $x \in \overline{A} \cap \operatorname{int} U$, which contradicts $U \cap A = \phi$. So we must have $U \cap A \neq \phi$.

Conversely, let every neighborhood of x intersect A. Suppose $x \notin \overline{A}$ per Definition 5.10. So there is at least one closed set F with $A \subseteq F$ but $x \notin F$. Then $x \in X\backslash F$. But $X\backslash F$ is open and in particular a neighborhood of x. However, $A \subseteq F$ implies $X\backslash F \cap A = \phi$—a contradiction to every neighborhood of x intersecting A. So indeed $x \in \overline{A}$.

ii) Suppose $x \in A \cup A'$. Then, by the definition of cluster points, every neighborhood of x intersects A. Thus, by i) $x \in \overline{A}$. Conversely, let $x \in \overline{A}$. If $x \in A$, then clearly $x \in A \cup A'$. Let $x \notin A$. Let U be a neighborhood of x. By i) $U \cap A \neq \phi$. But $x \notin A$. So $U \cap A\backslash\{x\} \neq \phi$. So $x \in A'$. So $\overline{A} \subseteq A \cup A'$. Hence $\overline{A} = A \cup A'$. ☐

What is the generalization of open balls to general topological spaces? The following is one:

Definition 5.13. Let (X, τ) be a topological space. A **base** for τ is a subcollection $\mathcal{B} \subseteq \tau$, such that every member of τ is a union of some members of $\mathcal{B}$.

Remark 5.14. It is evident that $\mathcal{B}$ is a base for (X, τ) provided $\mathcal{B}$ is a collection of open sets with the following property: $U \subseteq X$ is open if and only if for all $x \in U$, there exists some $B_x \in \mathcal{B}$ with $x \in B_x \subseteq U$.

Example 5.15. i) τ itself is a base for (X, τ);

ii) If X is a metric space, then $\mathcal{B} = \{B(x, \epsilon) : x \in X, \epsilon > 0\}$ is a base;

iii) $X = \mathbf{R}$ with the Euclidean topology has $\mathcal{B} = \{(a,b) : a,b \in \mathbf{Q}\}$ as a base;

iv) One can specify a topology by just specifying a base for it. For instance, let $X = \mathbf{R}$ and let $\mathcal{B} = \{(a,b) : a,b \in \mathbf{Q}\} \cup \{\{x\} : x \in \mathbf{R}\backslash\mathbf{Q}\}$. Then the open sets of this topology are $\tau = \{U \cup J :$ U is open in the usual topology of $\mathbf{R}$, and $J \subseteq X\backslash Q\}$. The real line $\mathbf{R}$, under this topology, is said to be the **Michael line**, and we occasionally write $\mathbf{R}$ as $\mathbf{R}_M$ to remind ourselves that this $\mathbf{R}$ is without the Euclidean topology. (We could also allow (a,b) with $a,b \in \mathbf{R}$ or $a,b \in \mathbf{R}\backslash\mathbf{Q}$ in the first half of this basis.)

Note (a,b) with $a,b \in \mathbf{R}\backslash\mathbf{Q}$ is both open and closed in this topology. It is open, since (a,b) is an open interval, and all open intervals are open in $\mathbf{R}_M$. It is closed since $\mathbf{R}_M\backslash(a,b) = (-\infty,a] \cup [b,\infty)$ and $(-\infty,a] = (-\infty,a) \cup \{a\}$ and $[b,\infty) = \{b\} \cup (b,\infty)$, i.e., $\mathbf{R}_M\backslash(a,b)$ is the union of four open sets. Thus $\mathbf{R}_M\backslash(a,b)$ is open, and so (a,b) is closed.

When is a collection of subsets of X a base for *some* topology on X? The following theorem provides an answer:

Theorem 5.16. *A family $\mathcal{B}$ of subsets of X is a base for* some *topology on X if and only if: i) $\bigcup_{B\in\mathcal{B}} B = X$, and ii) if $U,V \in \mathcal{B}$ and $x \in U \cap V$, then there exists $W \in \mathcal{B}$ such that $x \in W \subseteq U \cap V$.*

Example 5.17. [2] i) Let (X,d) be a metric space. Let $\mathcal{B} = \{B(x,\epsilon) : x \in X, \epsilon > 0\}$. Clearly $X = \bigcup_{B\in\mathcal{B}} B$. If $B(x,\epsilon_1) \cap B(y,\epsilon_2) \neq \phi$ and $z \in B(x,\epsilon_1) \cap B(y,\epsilon_2)$, let $\eta = \min\{\epsilon_1 - d(x,z), \epsilon_2 - d(y,z)\}$. Thus $\eta > 0$. Consider $B(z,\eta)$. Clearly $B(z,\eta) \in \mathcal{B}$ and $z \in B(z,\eta)$. Now $B(z,\eta) \subseteq B(x,\epsilon_1) \cap B(y,\epsilon_2)$. Indeed, if $w \in B(z,\eta)$, then $d(w,x) \leq d(w,z) + d(z,x) < \epsilon_1 - d(x,z) + d(z,x) = \epsilon_1$. So $w \in B(x,\epsilon_1)$. Similarly, $d(w,y) \leq d(w,z) + d(z,y) < \epsilon_2 - d(y,z) + d(y,z) = \epsilon_2$. So $w \in B(y,\epsilon_2)$, and thus $B(z,\eta) \subseteq B(x,\epsilon_1) \cap B(y,\epsilon_2)$.

ii) Let $X = \mathbf{R}$ and let $\mathcal{B} = \{[a,b) : a < b\}$. Clearly $\mathcal{B}$ satisfies both requirements of Theorem 5.14, so it is a base for some topology on X. $\mathbf{R}$ with this topology is called the **Sorgenfrey line** and is sometimes denoted $\mathbf{R}_S$.

Remark 5.18. Theorem 5.14 can also be used to determine if a given $\mathcal{B}$ is a base for an *a priori* specified topology τ. We would, then, also require the elements of $\mathcal{B}$ to be open sets in τ.

Definition 5.19. i) (Neighborhood basis) Let (X,τ) be a topological space, and let $x \in X$. A **neighborhood base** at x is a collection $\mathcal{B}_x$ of neighborhoods of x, such that if U is a neighborhood of x, then there exists $B \in \mathcal{B}_x$ with $B \subseteq U$.

ii) (Local base) A **local base** is a neighborhood base, all of whose members are open neighborhoods.

Remark 5.20. i) Every local base is a neighborhood base. ii) If $\mathcal{B}_x$ is a neighborhood base, then $\{\operatorname{int} B : B \in \mathcal{B}_x\}$ is a local base at x. iii) If $\mathcal{B}_x$ is a local base for all $x \in X$, then $\mathcal{B} = \bigcup_{x\in X} \mathcal{B}_x$ is a base for X. iv) If $\mathcal{B}$ is a base for X, then for all $x \in X$, $\mathcal{B}_x = \{B \in \mathcal{B} : x \in B\}$ is a local base at x.

Interlude. We now present, mostly without proofs, the notions of countability and separability.

Definition 5.21. i) If (X,τ) has a countable neighborhood base at each $x \in X$, we say X is **first countable**. ii) If (X,τ) has a countable base, we say X is **second countable**.

(In the above definition, what is meant is each member of the said base is countable.)

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[2] I diverge again from the handouts numbering here.

Remark 5.22. In Definition 5.19, we could replace "neighborhood base" by "local base". (Why? See Remark 4.18-i and -ii.)

Exercise 5.23. Show a second countable space is first countable. (Use Remark 4.18-iv.)

Exercise 5.24. i) $\mathbf{R}_M$ is first countable but not second countable. ii) An uncountable space with the co-finite topology is not first countable.

Definition 5.25 (Separability)**.** Topological space (X, τ) is said to be **separable** if X has a countable dense set.

Example 5.26. i) $\mathbf{R}$ is separable since $\mathbf{Q}$ is dense in $\mathbf{R}$;

ii) $\mathbf{R}_S$ is also separable since $\mathbf{Q}$ is dense in the Sorgenfrey line. (Why? Indeed every open set in $\mathbf{R}_S$ is a union of intervals of the form $[a, b)$, and each such interval intersects $\mathbf{Q}$.)

iii) $\mathbf{R}_M$ is not separable, since any dense subset of $\mathbf{R}_M$ must contain all irrationals, and thus must be uncountable. Indeed, each $x \in \mathbf{R}\backslash\mathbf{Q}$ is open in the topology of $\mathbf{R}_M$, and thus a dense set must "intersect" each $x \in \mathbf{R}\backslash\mathbf{Q}$, i.e., contain each $x \in \mathbf{R}\backslash\mathbf{Q}$.

Next we summarize relations between I and II countability and separability.

Theorem 5.27. *i) Every II countable space is separable. ii) The converse is not true. The Sorgenfrey line is separable but not II countable. iii) Every separable* metric *space is II countable.*

Definition 5.28. i) A collection of subsets of X, such that X is the union of the members of this collection, is said to be a **cover** of X. ii) A cover whose members are open is said to be an **open cover** of X. iii) If U is a cover and $V \subseteq U$ is also a cover, we say V is a **subcover** of X.

Definition 5.29 (Lindolöf space)**.** If (X, τ) is such that every open cover of X has a countable subcover, we say X is a **Lindelöf space**.

Theorem 5.30. *Let X be a* metric *space. Then the following are equivalent: i) X is II countable; ii) X is separable; iii) X is Lindelöf.*